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Intrinsic degree of coherence of classical and quantum states

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In the context of the two-dimensional (2D) polarization states of light, the degree of polarization P_2 is equal to the maximum value of the degree of coherence over all possible bases. Therefore, P_2 can be referred to as the intrinsic degree of coherence of a 2D state. In addition to (i) the maximum degree of coherence interpretation, P_2 also has the following interpretations: (ii) it is the Frobenius distance between the state and the maximally incoherent identity state, (iii) it is the norm of the Bloch vector representing the state, (iv) it is the distance to the center of mass in a configuration of point masses with magnitudes equal to the eigenvalues of the state, (v) it is the visibility in a polarization interference experiment, and (vi) it is the weightage of the pure part of the state. Among these six interpretations of P_2, the Bloch vector norm, Frobenius distance, and center-of-mass interpretations have previously been generalized to derive an analogous basis-independent measure P_N for N-dimensional (ND) states. In this paper, by extending the concepts of visibility, degree of coherence, and weightage of the pure part to ND spaces, we show that these three remaining interpretations of P_2 also generalize to the same quantity P_N, establishing P_N as the intrinsic degree of coherence of ND states. We then extend P_N to the $N \to \infty$ limit to quantify the intrinsic degree of coherence P_∞ of infinite-dimensional states in the orbital angular momentum, photon number, and position-momentum degrees of freedom.

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1. INTRODUCTION

Coherence is the physical property responsible for interference phenomena observed in nature and is the subject matter of the classical and quantum theories of coherence [1–7]. Both these highly successful theories quantify coherence in terms of the visibility or contrast of the interference. The key difference is that whereas the classical theory formulates the visibility in terms of correlation functions involving products of field amplitudes [2–4], the quantum theory of optical coherence employs correlation functions involving products of field operators that in general may not commute [5–7]. In comparison to the classical theory that fails to explain the higher-order correlations of certain quantum light fields [8,9], the quantum theory can be used to quantify the correlations of a general light field to arbitrary orders. However, as far as effects arising from second-order correlations of light fields are concerned, the classical and quantum theories have identical predictions implying that both can be interchangeably used.

For quantifying second-order correlations, the quantity of central interest is the degree of coherence, which is just the suitably normalized second-order correlation function involving electromagnetic fields at two distinct spacetime points or polarization directions [2,3]. In the context of a partially polarized field represented by a 2×2 polarization matrix ρ, the degree of coherence is the magnitude of the suitably normalized off-diagonal entry that quantifies the correlations between the field components along a specific pair of orthogonal polarizations. Thus, the degree of coherence is a manifestly basis-dependent measure of coherence. In contrast, the maximum degree of coherence over all possible orthonormal polarization bases is a basis-independent measure of coherence known as the degree of polarization [4]. Owing to this basis-independent maximum degree of coherence interpretation, in this paper we also refer to the degree of polarization P_2 as the "intrinsic degree of coherence" of the field. For the polarization matrix ρ, which is normalized, P_2 is given by

$$P_2 = \sqrt{2\,\mathrm{Tr}\,(\rho^2) - 1}. \tag{1}$$

In addition to (i) the maximum degree of coherence interpretation, P_2 also has the following interpretations [2]: (ii) it is the norm of the Bloch vector representing the state, (iii) it is the Frobenius distance between the state and the completely incoherent state [10], (iv) it is the distance to the center of mass in a configuration of point masses of magnitudes equal to the eigenvalues of the state [11], (v) it is the visibility obtained in a polarization interference experiment, and (vi) it is the weightage of the completely polarized part of the state. These six interpretations together provide a mathematically appealing and physically intuitive quantification of the intrinsic polarization correlations of a field in a basis-independent manner.

While the need for a basis-independent quantification of coherence has been recognized long ago in both classical and quantum theories of optical coherence, such a quantification has fully been achieved only for the two-dimensional (2D) polarization states of light. In this context, it is known that the 2×2 polarization matrix describing the polarization state of a classical light field is formally identical to the 2×2 density matrix describing a quantum two-level system. Moreover, there is a one-to-one correspondence between the Poincare sphere representation of partially polarized fields in terms of Stokes parameters [12] and the Bloch sphere representation of qubits in terms of the Bloch vector components [13]. By this correspondence, the measure P_2 encodes essentially the same information as the quantum purity, and can therefore be used to quantify the intrinsic coherence of both classical and quantum 2D states [14]. However, a generalized coherence measure analogous to P_2 that retains all its interpretations has not been obtained for higher-dimensional states so far.

For quantifying the coherence of higher-dimensional systems, a number of studies in recent years have taken a resource theoretic approach [15–20]. However, the present paper does not follow this resource theoretic approach. Instead, it follows an approach from optical coherence theory, which seeks to generalize the basis-independent measure of coherence P_2 and all its known interpretations to quantify the intrinsic degree of coherence of higher-dimensional classical and quantum states.

The first efforts in generalizing P_2 to higher dimensions were carried out by Barakat [21,22] and Samson and Olson [23,24]. In these efforts, they derived a basis-independent measure P_N for an $N \times N$ polarization matrix ρ by generalizing the Bloch vector norm interpretation of P_2 to an N-dimensional (ND) space. In particular, they showed that for a normalized ρ,

$$P_N = \sqrt{\frac{N\mathrm{Tr}(\rho^2) - 1}{N - 1}}. \tag{2}$$

Recently, following up on previous generalizations for 3D [25] and 4D [26] spaces, the Frobenius distance interpretation of P_2 [10] was generalized to ND spaces to also yield P_N [27]. In addition, the center-of mass-interpretation when applied to ND states yields P_N as the generalized measure. Thus, it has so far been possible to show that P_N has three of the six interpretations of P_2. However, the generalization of the remaining three interpretations has either not been attempted or has had limited success [14,28,29]. In this paper, we take up the other three interpretations of P_2, namely, the visibility, degree of coherence, and weightage of the pure part interpretations and extend them to ND spaces. We show that even these three interpretations of P_2 generalize to the same measure P_N. In essence, by demonstrating that P_N has all six interpretations of P_2, we theoretically establish P_N as quantifying the intrinsic degree of coherence of ND states. We then extend P_N to the $N \to \infty$ limit to quantify the intrinsic degree of coherence P_∞ of infinite-dimensional states.

The paper is organized as follows. In Section 2, we present a conceptual description of the degree of polarization. In Section 3, we describe the existing work on how the expression for P_N is obtained by generalizing the Bloch vector norm, Frobenius distance, and center-of-mass interpretations of P_2 to ND states. In Section 4, we generalize the concepts of visibility, degree of coherence, and weightage of the pure part to ND spaces, demonstrate that each of these interpretations of P_2 uniquely generalizes to P_N, and thereby establish P_N as the intrinsic degree of coherence of finite ND classical and quantum states. In Section 5, we consider infinite-dimensional states in the orbital angular momentum (OAM), photon number, and position and momentum bases, and show that the intrinsic degree of coherence P_∞ of a normalizable state ρ is given by $P_\infty = \sqrt{\mathrm{Tr}(\rho^2)}$. In the rest of the paper, we will use the symbol ρ to denote the density matrix of dimensionality 2, N, or ∞ depending on the context. Also, we will denote the $N \times N$ identity matrix by $\mathbb{1}_N$.

2. DEGREE OF POLARIZATION

The polarization state of an electromagnetic field can be represented by a positive-semidefinite 2×2 Hermitian matrix. It is referred to as the polarization matrix or the coherence matrix and is defined as [1]

$$\rho = \begin{bmatrix} \langle E_1 E_1^* \rangle & \langle E_1 E_2^* \rangle \\ \langle E_1^* E_2 \rangle & \langle E_2 E_2^* \rangle \end{bmatrix} = \begin{bmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{bmatrix}. \tag{3}$$

Here, $\langle \cdots \rangle$ denotes the ensemble average over many realizations of the field, and E_1 and E_2 denote the electric field components along two mutually orthonormal polarization directions represented by the basis vectors $\{|1\rangle, |2\rangle\}$, and ρ_{ij} with $i, j = 1, 2$ denoting the matrix elements of ρ in the $\{|1\rangle, |2\rangle\}$ basis. The basis-dependent quantity $\mu_2 = |\rho_{12}|/\sqrt{\rho_{11}\rho_{22}}$ is called the degree of coherence between the polarization basis vectors $\{|1\rangle$ and $|2\rangle\}$. It was shown by Wolf in a classic paper that the maximum value of μ_2 over all possible choices of the bases in the 2D Hilbert space is equal to the degree of polarization P_2 [4], which for a normalized ρ, can be shown to be [2]

$$P_2 = \sqrt{1 - 4\det\rho} = \sqrt{2\,\mathrm{Tr}\,(\rho^2) - 1}. \tag{4}$$

As the trace and the determinant are invariant under unitary operations, P_2 is a basis-independent quantity. Furthermore, $0 \le P_2 \le 1$ with $P_2 = 1$ only when ρ is a perfectly polarized field (pure state) and $P_2 = 0$ only when ρ is the completely unpolarized field (completely mixed state) represented by the identity matrix. In the next two sections, we consider the six known interpretations of P_2 that justify its suitability as an intrinsic degree of coherence for 2D states. Following a brief description of each interpretation, we present the generalization to ND space and obtain P_N as the ND analog of P_2.

3. EXISTING WORKS ON GENERALIZING INTERPRETATIONS OF P_2 TO ND STATES

A. Bloch Vector Norm Interpretation

1. 2D States

It is known that an arbitrary 2D state ρ has the following unique decomposition in terms of the Stokes parameters [30]:

$$\rho = \frac{1}{2}\left(\mathbb{1}_2 + \sum_{i=1}^{3} r_i \sigma_i\right). \tag{5}$$

Here, σ_1, σ_2, and σ_3 are the Pauli matrices, and the real scalar quantities r_i's are called the Stokes parameters of the state. Such a parametrization is possible due to the fact that σ_i's, which are the generators of the Lie group SU(2), form an orthonormal basis in the real vector space of traceless 2×2 Hermitian matrices with respect to the Hilbert–Schmidt inner product, $(A, B) \equiv \mathrm{Tr}(A^\dagger B)$. Consequently, the parameters r_i can be regarded as the components of a 3D vector $\boldsymbol{r} \equiv (r_1, r_2, r_3)$, which is referred to as the Bloch vector representing the state in this vector space. For a 2D density matrix ρ, the condition $\mathrm{Tr}\rho^2 \le 1$ is both necessary and sufficient to ensure positive-semidefiniteness, which in turn implies that the space of physical states is characterized by $0 \le |\boldsymbol{r}| \le 1$. This space can be imagined to be a closed sphere in three dimensions, termed as the Bloch sphere. The pure states reside on the surface of this sphere with $|\boldsymbol{r}| = 1$, whereas the maximally incoherent state $\mathbb{1}_2/2$ with $|\boldsymbol{r}| = 0$ resides at the center. From Eq. (4), it can be shown that the norm of the Bloch vector is equal to P_2, i.e., $|\boldsymbol{r}| = \sqrt{\sum_{i=1}^{3} |r_i|^2} = P_2$ [30]. This way, P_2 is interpreted as the norm of the Bloch vector representing the state.

2. ND States

In direct correspondence with Eq. (5), it has been shown that any ND state ρ can be decomposed as [31–34]

$$\rho = \frac{1}{N}\left(\mathbb{1}_N + \sqrt{\frac{N(N-1)}{2}} \sum_{i=1}^{(N^2-1)} r_i \Lambda_i\right), \tag{6}$$

where Λ_i's are the generalized $N \times N$ Gellmann matrices, and the scalar quantities r_i's are the ND analogs of Stokes parameters. In exact analogy with the 2D case, this parametrization is made possible by the fact that Λ_i's, which are the $(N^2 - 1)$ generators of the Lie group SU(N), form an orthonormal basis in the real vector space of traceless $N \times N$ Hermitian matrices with respect to the Hilbert–Schmidt inner product. The parameters r_i form the components of the $(N^2 - 1)$-dimensional Bloch vector $\boldsymbol{r}$ representing the state ρ. We note that in contrast to the 2D case, the condition $\mathrm{Tr}\rho^2 \le 1$ is not sufficient to ensure positive-semidefiniteness of ND density matrices. Consequently, only a subset of states represented by the $(N^2 - 1)$-dimensional sphere and defined by $0 \le |\boldsymbol{r}| \le 1$ corresponds to physical states [32,33].

Barakat [21,22] and Samson and Olson [23,24] were the first ones to show that the norm of the ND Bloch vector is the degree of polarization P_N of the state. The derivations of P_N by both Barakat [21,22] and Samson and Olson [23,24] were presented in terms of the eigenvalues of ρ and not in terms of the Gellman matrices. For 3D states, an explicit derivation of P_3 in terms of 3D Gellman matrices was carried out by Setälä *et al.* [35,36] who also demonstrated usefulness of P_3 for studying optical near fields and evanescent fields. We now present the derivation for ND states explicitly in term of ND Gellman matrices and obtain the expression of P_N as in Eq. (2).

We note that the set of $(N^2 - 1)$ generalized Gellmann matrices Λ_i's of Eq. (6) comprises three subsets: the set $\{U\}$ of $N(N-1)/2$ symmetric matrices, the set $\{V\}$ of $N(N-1)/2$ anti-symmetric matrices, and the set $\{W\}$ of $(N-1)$ diagonal matrices. The explicit forms of these matrices in the orthonormal basis $\{|i\rangle\}_{i=1}^{N}$, where $|i\rangle$ is an ND column vector with the i^{th} entry being 1 and others being 0, are given by [32]

$$U_{jk} = |j\rangle\langle k| + |k\rangle\langle j|, \qquad V_{jk} = -i|j\rangle\langle k| + i|k\rangle\langle j|,$$
$$\text{and} \quad W_l = \sqrt{\frac{2}{l(l+1)}}\left(\sum_{m=1}^{l} |m\rangle\langle m| - l|l+1\rangle\langle l+1|\right), \tag{7}$$

where $1 \le j < k \le N$ and $1 \le l \le (N-1)$. In terms of these definitions, we write Eq. (6) as

$$\rho = \frac{1}{N}\left[\mathbb{1}_N + \sqrt{\frac{N(N-1)}{2}} \times \left(\sum_{j=1}^{N}\sum_{k=j+1}^{N}\{u_{jk}U_{jk} + v_{jk}V_{jk}\} + \sum_{l=1}^{N-1} w_l W_l\right)\right], \tag{8}$$

where u_{jk}, v_{jk}, and w_l are the Bloch vector components along the Gellmann matrices U_{jk}, V_{jk}, and W_l, respectively. Here, we have relabeled the set of components $\{r_i\}$ and the set of matrices $\{\Lambda_i\}$ of Eq. (6) by the set of parameters $\{\{u_{jk}\}, \{v_{jk}\}, \{w_l\}\}$ and the set of matrices $\{\{U_{jk}\}, \{V_{jk}\}, \{W_l\}\}$, respectively. We calculate the components u_{jk}, v_{jk}, and w_l in terms of the density matrix elements and find them to be

$$u_{jk} = \sqrt{\frac{N}{2(N-1)}}(\rho_{jk}+\rho_{kj}), \quad v_{jk} = i\sqrt{\frac{N}{2(N-1)}}(\rho_{jk}-\rho_{kj}),$$
$$w_l = \sqrt{\frac{N}{l(l+1)(N-1)}}\left(\sum_{m=1}^{l} \rho_{mm} - l\rho_{l+1,l+1}\right). \tag{9}$$

The norm of the Bloch vector $\boldsymbol{r}$ defined as $|\boldsymbol{r}| = \sqrt{\sum_{i=1}^{(N^2-1)} r_i^2}$ is therefore given by

$$|\boldsymbol{r}| = \sqrt{\sum_{j=1}^{N}\sum_{k=j+1}^{N}[u_{jk}^2 + v_{jk}^2] + \sum_{l=1}^{N-1} w_l^2}. \tag{10}$$

In order to evaluate $|\boldsymbol{r}|$, we first find that

$$\sum_{j=1}^{N}\sum_{k=j+1}^{N}[u_{jk}^2 + v_{jk}^2] = \frac{2N}{N-1}\sum_{j=1}^{N}\sum_{k=j+1}^{N}|\rho_{jk}|^2. \tag{11}$$

We then evaluate the other summation in Eq. (10) to be

$$\sum_{l=1}^{N-1} w_l^2 = \sum_{l=1}^{N-1} \frac{N}{l(l+1)(N-1)} \left(\sum_{m=1}^{l} \rho_{mm} - l\rho_{l+1,l+1} \right)^2$$
$$= \frac{N}{N-1} \left[\sum_{i=1}^{N} \rho_{ii}^2 \left\{ \sum_{j=i}^{N-1} \frac{1}{j(j+1)} + \frac{i-1}{i} \right\} - \frac{2}{N} \sum_{i=1}^{N} \sum_{j=i+1}^{N} \rho_{ii}\rho_{jj} \right]$$
$$= \sum_{i=1}^{N} \rho_{ii}^2 - \frac{2}{N-1} \sum_{i=1}^{N} \sum_{j=i+1}^{N} \rho_{ii}\rho_{jj}. \quad \textbf{(12)}$$

By substituting Eqs. (11) and (12) into Eq. (10), we obtain

$$|\boldsymbol{r}| = P_N = \sqrt{\frac{N\mathrm{Tr}(\rho^2) - 1}{N-1}} = P_N. \quad \textbf{(13)}$$

Thus P_N, like its 2D analog, can be interpreted as the norm of the Bloch vector corresponding to the ND state.

B. Frobenius Distance Interpretation

1. 2D States

For a 2D state ρ, it was known that the degree of polarization P_2 can be viewed as the Frobenius distance between the state ρ and the completely incoherent state $\mathbb{1}_2/2$ [10], i.e.,

$$P_2 = \sqrt{2} \left\| \rho - \frac{\mathbb{1}_2}{2} \right\|_F = \sqrt{2\,\mathrm{Tr}\,(\rho^2) - 1}. \quad \textbf{(14)}$$

Here, the Frobenius distance is quantified using the Frobenius norm, defined as $\|A\|_F \equiv \sqrt{\mathrm{Tr}(A^\dagger A)}$, with the normalization factor ensuring that $0 \leq P_2 \leq 1$. We see that the expressions of P_2 in Eqs. (4) and (14) are the same.

2. ND States

The Frobenius distance interpretation was first generalized to 3D [25] and 4D [26] states by Luis. More recently, Yao *et al.* [27] have generalized the Frobenius distance interpretation to ND states to define P_N as

$$P_N \equiv \sqrt{\frac{N}{N-1}} \left\| \rho - \frac{\mathbb{1}_N}{N} \right\|_F = P_N = \sqrt{\frac{N\mathrm{Tr}(\rho^2) - 1}{N-1}}. \quad \textbf{(15)}$$

In other words, P_N is the Frobenius distance between the state ρ and the completely incoherent state $\mathbb{1}_N/N$ in the space of $N \times N$ density matrices. The normalization factor in Eq. (15) is again chosen such that $0 \leqslant P_N \leqslant 1$. We note that the expressions of P_N in Eqs. (2) and (15) are the same. Furthermore, it can be verified that when ρ is pure, $\mathrm{Tr}(\rho^2) = 1$, implying $P_N = 1$, whereas when $\rho = \mathbb{1}_N/N$, $\mathrm{Tr}(\rho^2) = 1/N$, implying $P_N = 0$.

C. Center-of-Mass Interpretation

In a recent study, Alonso *et al.* [11] have discussed a geometric interpretation of the measure P_N of Eq. (2) as the distance to the center of mass in a configuration of point masses.

1. 2D States

Consider a configuration of two point masses of magnitudes equal to the eigenvalues λ_1 and λ_2 of the state, each placed at a unit distance from the origin in opposite directions in a 1D Euclidean space. The distance Q to the center of mass of this configuration from the origin is given by

$$Q = \left| \frac{\lambda_1 - \lambda_2}{\lambda_1 + \lambda_2} \right| = P_2. \quad \textbf{(16)}$$

Thus, P_2 has the interpretation as the distance of the center of mass from the origin in this configuration.

2. ND States

Consider a configuration of N point masses of magnitudes equal to the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_N$ of ρ, each placed at a unit distance from the origin and equally spaced from one another such that they constitute a regular $(N-1)$-simplex in an $(N-1)$-dimensional Euclidean space. The distance Q to the center of mass of this configuration is given by

$$Q = \sqrt{\frac{\sum_{i=1}^{N-1} \sum_{j=i+1}^{N} (\lambda_i - \lambda_j)^2}{(N-1)\left(\sum_{i=1}^{N} \lambda_i \right)^2}} = P_N. \quad \textbf{(17)}$$

Therefore, P_N is equal to the distance of the center of mass of this configuration from the origin.

4. GENERALIZING OTHER INTERPRETATIONS OF P_2 TO ND STATES

A. Maximum Degree of Coherence Interpretation

1. 2D States

As pointed out in Section 2 in the context of 2D polarization sates, the basis-dependent quantity μ_2 in Eq. (3) quantifies the degree of coherence between the mutually orthogonal polarization states represented by $|1\rangle$ and $|2\rangle$. Using Eqs. (4) and (19), it can be shown that $0 \leq \mu_2 \leq P_2$ and also that μ_2 attains the maximum value P_2 when the basis $\{|1\rangle, |2\rangle\}$ is such that $\rho_{11} = \rho_{22}$ [2,4], i.e.,

$$\max_{\{|1\rangle,|2\rangle\} \in \mathbb{S}} \mu_2 = P_2. \quad \textbf{(18)}$$

In this way, P_2 is interpreted as the maximum of μ_2 over the set $\mathbb{S}$ of all orthonormal bases in the 2D Hilbert space. In order to generalize the definition of the degree of coherence for ND states, we rewrite μ_2 as

$$\mu_2 = \sqrt{\frac{|\rho_{12}|^2}{\rho_{11}\rho_{22}}}. \quad \textbf{(19)}$$

We find that while the numerator $|\rho_{12}|^2$ quantifies the correlation between the basis vectors $|1\rangle$ and $|2\rangle$, the denominator provides the normalization such that $0 \le \mu_2 \le 1$. Our aim is to define an ND degree of coherence μ_N such that it reduces to μ_2 for $N = 2$ and lies between 0 and 1.

2. ND States

We use the definition in Eq. (19) to generalize the concept of the degree of coherence to ND states. We expect the generalized quantity μ_N to be basis dependent, the maximum of which must be equal to the ND intrinsic degree of coherence P_N. Therefore, in analogy with the definition of μ_2 in Eq. (19), we define the ND degree of coherence μ_N as

$$\mu_N = \sqrt{\frac{\sum_{i=1}^{N-1}\sum_{j=i+1}^{N}|\rho_{ij}|^2}{\sum_{i=1}^{N-1}\sum_{j=i+1}^{N}\rho_{ii}\rho_{jj}}}. \tag{20}$$

Here, ρ_{ij} are the matrix elements of the state ρ in an orthonormal basis $\{|1\rangle, |2\rangle, \ldots, |N\rangle\}$. The numerator is the sum of the squared magnitudes of all the off-diagonal terms, and the denominator is the sum of the products of the pairs of diagonal terms. As expected, μ_N as defined above reduces to μ_2 for $N = 2$, and the normalization term in the denominator makes sure that μ_N lies between 0 and 1. We further note that μ_N is a basis-dependent quantity. Now, in order for μ_N to be considered as the ND analog of μ_2, we need to show that the maximum value of μ_N over the set of all possible ND bases is equal to P_N. From Eqs. (15) and (20), we have

$$\mu_N^2 = \frac{\sum_{i=1}^{N-1}\sum_{j=i+1}^{N}|\rho_{ij}|^2}{\sum_{i=1}^{N-1}\sum_{j=i+1}^{N}\rho_{ii}\rho_{jj}} = \frac{\frac{1}{2}\left(\sum_{i=1}^{N}\sum_{j=1}^{N}|\rho_{ij}|^2 - \sum_{i=1}^{N}\rho_{ii}^2\right)}{\frac{1}{2}\left(\sum_{i=1}^{N}\sum_{j=1}^{N}\rho_{ii}\rho_{jj} - \sum_{i=1}^{N}\rho_{ii}^2\right)}$$

$$= \frac{\mathrm{Tr}(\rho^2) - \sum_{i=1}^{N}\rho_{ii}^2}{1 - \sum_{i=1}^{N}\rho_{ii}^2} = 1 - \frac{1 - \mathrm{Tr}(\rho^2)}{1 - \sum_{i=1}^{N}\rho_{ii}^2}. \tag{21}$$

From the above equation, it is clear that μ_N^2 attains its minimum value when the sum $\sum_{i=1}^{N}\rho_{ii}^2$ is maximum. The sum is maximum when ρ_{ii} is equal to 1 only for a particular i and is zero for the rest, in which case the sum $\sum_{i=1}^{N}\rho_{ii}^2 = \mathrm{Tr}(\rho^2)$, implying $\min \mu_N = 0$. Furthermore, μ_N^2 attains its maximum value when the sum $\sum_{i=1}^{N}\rho_{ii}^2$ is minimum. It is straightforward to show that the sum $\sum_{i=1}^{N}\rho_{ii}^2$ is minimum when $\rho_{11} = \rho_{22} = \cdots = \rho_{NN} = 1/N$, in which case $\sum_{i=1}^{N}\rho_{ii}^2 = \sum_{i=1}^{N}(1/N)^2 = 1/N$. Therefore, from Eq. (21), we have

$$\max_{\{|1\rangle,|2\rangle,\ldots|N\rangle\}\in\mathbb{S}} \mu_N = \sqrt{\frac{N\,\mathrm{Tr}(\rho^2) - 1}{N-1}} = P_N, \tag{22}$$

which is in direct correspondence with Eq. (18). Thus, as in the 2D case, we find that the maximum of μ_N over the set $\mathbb{S}$ of all orthonormal bases in the ND Hilbert space is equal to the intrinsic degree of coherence P_N. Moreover, the maximum is achieved in the basis where all the diagonal entries are equal, again as is true in the 2D case. While our analysis does not present a clear physical reasoning for defining μ_N as Eq. (20), the fact that μ_N satisfies all the mathematical properties of μ_2 strongly suggests that μ_N is the ND analog of μ_2, and can therefore be referred to as the ND degree of coherence.

We now note that our above analysis is physically distinct from a recent study [19] that relates the maximal resource theoretic coherence of a state over unitary transformations to the state purity. The distinction arises because whereas optical coherence theory quantifies the system's ability to interfere, the resource theory of coherence quantifies the amount of superposition in a specific basis that can be exploited for certain quantum protocols. In order to illustrate this difference in the context of a 2D state ρ, we consider the l_1-norm measure $|\rho_{12}|$ from resource theory, and the degree of coherence $\mu_2 = |\rho_{12}|/\sqrt{\rho_{11}\rho_{22}}$ of Eq. (19) from optical coherence theory. For a pure state $\rho = |\psi\rangle\langle\psi|$, where $|\psi\rangle = \epsilon|1\rangle + \sqrt{1-\epsilon^2}|2\rangle$ with $\epsilon \to 0$, we have $|\rho_{12}| \to 0$, which implies that the state is incoherent in a resource theoretic sense, whereas $\mu_2 = 1$, which implies that the state is fully coherent in the optical coherence theoretic sense. Therefore, while it is interesting that similar relations between maximal coherence and purity hold in both theories, these relations are physically distinct.

B. Visibility Interpretation

1. 2D States

The visibility interpretation of P_2 for a 2D state was given by Wolf [4] using a polarization interference scheme (see Section 6.2 of Ref. [1]). As depicted in Fig. 1, we discuss this scheme with slight modifications in order to make it more amenable to generalization to higher dimensions. A field in the polarization state ρ, as given by Eq. (3), first passes through a wave plate (WP) that introduces a phase δ between the two mutually orthogonal directions represented by vectors $|1\rangle$ and $|2\rangle$. The field then passes through a rotation plate (RP) that rotates the polarization state by an angle θ. Finally, the field is detected using the polarizing beam splitter (PBS) in the two orthogonal polarization directions $|1\rangle$ and $|2\rangle$. The corresponding detection probabilities I_1 and I_2 at the two output ports are given by

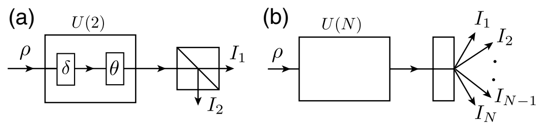

Fig. 1. (a) Schematic setup for describing degree of polarization P_2 as the visibility in a polarization interference experiment. (b) Schematic setup for describing N-dimensional degree of polarization or N-dimensional intrinsic degree of coherence P_N as the N-dimensional visibility in an interference experiment. PBS, polarizing beam splitter; NPS, N-port splitter.

$$I_1 = \rho_{11}\cos^2\theta + \rho_{22}\sin^2\theta + |\rho_{12}|\sin\theta\cos\theta\cos(\beta+\delta),$$
$$I_2 = \rho_{11}\sin^2\theta + \rho_{22}\cos^2\theta - |\rho_{12}|\sin\theta\cos\theta\cos(\beta+\delta),$$

where $\rho_{12} = |\rho_{12}|e^{i\beta}$. The visibility V of the interference pattern is defined as (see Section 6.2 of Ref. [1])

$$V = \frac{\langle I_1\rangle_{\max(\delta,\theta)} - \langle I_1\rangle_{\min(\delta,\theta)}}{\langle I_1\rangle_{\max(\delta,\theta)} + \langle I_1\rangle_{\min(\delta,\theta)}}, \quad (23)$$

where $\langle I_1\rangle_{\max(\delta,\theta)}$ and $\langle I_1\rangle_{\min(\delta,\theta)}$ are the maximum and minimum values of I_1, respectively, over all possible δ and θ. Similarly, we can equivalently define the visibility as

$$V = \max_{U\in U(2)}\left|\frac{I_1 - I_2}{I_1 + I_2}\right| = \max_{U\in U(2)} f(I_1, I_2), \quad (24)$$

where $U(2)$ is the group of 2D unitary matrices, and where we have denoted $|(I_1 - I_2)/(I_1 + I_2)|$ as $f(I_1, I_2)$ since we would find this notation to be more convenient when generalizing to ND spaces. The function $f(I_1, I_2)$ has the following properties: (i) it is 1 if and only if one among I_1 and I_2 is 1 and the other one is 0; (ii) it is 0 if and only if $I_1 = I_2$; (iii) it is a *Schur-convex* function, i.e., for two given sets of probabilities $\{I_1, I_2\}$ and $\{I_1', I_2'\}$, if $\{I_1', I_2'\}$ majorizes $\{I_1, I_2\}$, then $f(I_1, I_2) \le f(I_1', I_2')$ [37]. The maximization involved in Eq. (24) can be carried out using Schur's theorem, which states that the measured probability distribution of a state in any basis is majorized by the eigenvalue distribution of the state [38], i.e., $(I_1, I_2) \prec (\lambda_1, \lambda_2)$. Since there always exists a unitary transformation such that $I_1 = \lambda_1$ and $I_2 = \lambda_2$, $f(I_1, I_2)$ becomes maximum when $I_1 = \lambda_1$ and $I_2 = \lambda_2$, and in that case, we get

$$V = \max_{U\in U(2)} f(I_1, I_2) = f(\lambda_1, \lambda_2) = \left|\frac{\lambda_1 - \lambda_2}{\lambda_1 + \lambda_2}\right| = P_2, \quad (25)$$

i.e., P_2 equals the 2D visibility in a polarization interference experiment. The importance of the visibility interpretation is that it provides not only a physically intuitive way of understanding the degree of coherence but also an experimental scheme for measuring it.

2. ND States

In direct analogy with the scheme depicted in Fig. 1(a), Fig. 1(b) depicts the general interference situation for an ND density matrix ρ represented in an orthonormal basis $\{|1\rangle, |2\rangle, \dots, |N\rangle\}$. The density matrix ρ is acted upon by a general $N \times N$ unitary operator U, which can be realized by a combination of optical elements. The N-port splitter (NPS) divides the density matrix along N orthonormal states $\{|1\rangle, |2\rangle, \dots, |N\rangle\}$, and the detection probabilities along the basis vectors are represented by $\{I_1, I_2, \dots, I_N\}$. In analogy with the definition of $f(I_1, I_2)$ for the 2D case, we define

$$f(I_1, I_2, \dots, I_N) = \sqrt{\frac{\sum_{i=1}^{N-1}\sum_{j=i+1}^{N}(I_i - I_j)^2}{(N-1)\left(\sum_{i=1}^{N} I_i\right)^2}}, \quad (26)$$

which satisfies the following properties: (i) it is 1 if and only if $I_i = 1$ for some $i = k$, and $I_i = 0$ for $i \ne k$, where $i = 1, 2, \dots, N$ and $k \le N$; (ii) it is 0 if and only if all the probabilities are equal, i.e., $I_i = 1/N$, where $i = 1, 2, \dots, N$; (iii) it is a *Schur-convex* function, as may be proved using theorem II.3.14 of Ref. [37]. We know by virtue of Schur's theorem [38] that $\{I_1, I_2, \dots, I_N\} \prec \{\lambda_1, \lambda_2, \dots, \lambda_N\}$, where λ_is are eigenvalues of the density matrix. We also know that there always exists a unitary transformation $U \in U(N)$, such that $\{I_1, I_2, \dots, I_N\} = \{\lambda_1, \lambda_2, \dots, \lambda_N\}$. Using these facts, we define the ND visibility V as $f(I_1, I_2 \dots, I_N)$ maximized over $U(N)$, i.e.,

$$V = \max_{U\in U(N)} f(I_1, I_2 \dots, I_N) = \sqrt{\frac{\sum_{i=1}^{N-1}\sum_{j=i+1}^{N}(\lambda_i - \lambda_j)^2}{(N-1)\left(\sum_{i=1}^{N}\lambda_i\right)^2}}$$
$$= \sqrt{\frac{N\,\mathrm{Tr}(\rho^2) - 1}{N-1}} = P_N. \quad (27)$$

Thus, we find that just as in the 2D case, P_N has the interpretation as the ND visibility of an experiment.

C. Weightage of Pure Part Interpretation

1. 2D States

In the context of partially polarized fields, it has been shown that any 2D polarization state ρ can be uniquely decomposed into a weighted mixture of two fields, one of which is completely polarized or pure, and the other one completely unpolarized or fully mixed [1,2]. Mathematically, this implies that

$$\rho = s_1|\psi_1\rangle\langle\psi_1| + (1 - s_1)\frac{\mathbb{1}_2}{2}, \quad (28)$$

where $|\psi_1\rangle$ represents the completely polarized pure state, $s_1 = \lambda_1 - \lambda_2$ with λ_1 and λ_2 being the eigenvalues of ρ denotes the weightage of the pure part, and $\mathbb{1}_2$ is the completely unpolarized state. From Eq. (25), we know that for a normalized ρ, $\lambda_1 - \lambda_2 = P_2$, from which we get

$$s_1 = \lambda_1 - \lambda_2 = P_2. \quad (29)$$

In other words, P_2 is equal to the weightage of the pure portion of the state. This interpretation is physically intuitive as it implies that in order to prepare the state by mixing together a pure state and the completely mixed state, the needed weightage of the pure part is P_2.

2. ND States

We now generalize this interpretation of P_2 to higher dimensions. The quantification of P_2 in terms of the weightage of its pure part is possible only because of the existence of the unique decomposition in Eq. (28). However, it is now known that such a unique decomposition in terms of just two matrices is not possible for ND states [39–41]. For a 3D polarization state, it has been shown that a unique decomposition is possible in terms of three matrices, one of which is the rank-1 matrix, which is a pure state, the second one is a rank-2 matrix, and the third one is the identity matrix [28]. It has been argued that the weightage of the pure part of this decomposition, which is equal to $\lambda_1 - \lambda_2$, where λ_1 and λ_2 are the two largest eigenvalues of ρ, could be taken as the degree of polarization of the 3D state. However, a few issues have been pointed out regarding this decomposition because of which the weightage of the rank-1 matrix of this decomposition cannot in general be taken as the 3D degree of polarization [14,29].

In contrast, we now show that it is possible to have a unique decomposition of an ND state as a weighted mixture of N matrices as given below, one of which is completely mixed and the rest $N-1$ are completely pure:

$$\rho = \sum_{i=1}^{N-1} s_i |\psi_i\rangle\langle\psi_i| + \left(1 - \sum_{i=1}^{N-1} s_i\right) \frac{\mathbb{1}_N}{N}. \tag{30}$$

Here, the states $\{|\psi_i\rangle\}$'s are pure and orthonormal, and the corresponding weightages s_i's are real and non-negative. In order to ensure a unique decomposition for every physical density matrix, it must be verified that the number of independent parameters are identical on the two sides of Eq. (30). On the left side, the density matrix ρ has (N^2-1) free parameters. On the right side, (i) there are $(N-1)s_i$'s, (ii) each of the $(N-1)|\psi_i\rangle$'s has $2(N-1)$ free parameters, and (iii) the mutual orthogonality between $|\psi_i\rangle$'s would introduce $(N-1)(N-2)$ constraints. These conditions imply (N^2-1) free parameters on the right-hand side as well. We introduce an additional vector $|\psi_N\rangle$ to the set of $(N-1)|\psi_i\rangle$'s, such that $|\psi_i\rangle$ with $i = 1 \ldots N$ form an orthonormal and complete basis, i.e., $\sum_{i=1}^{N} |\psi_i\rangle\langle\psi_i| = \mathbb{1}_N$. Now, if Eq. (30) is written in this $|\psi_i\rangle$ basis, then the right-hand side is completely diagonal. This implies that the representation of ρ on the left-hand side must also be diagonal in this basis, i.e., $|\psi_i\rangle$'s must necessarily be the eigenvectors of ρ with $\rho = \sum_{i=1}^{N} \lambda_i |\psi_i\rangle\langle\psi_i|$. Here, we have denoted the corresponding eigenvalues as λ_i and have assumed $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N$. Equation (30) therefore takes the form

$$\rho = \sum_{i=1}^{N-1} (\lambda_i - \lambda_N) |\psi_i\rangle\langle\psi_i| + (N\lambda_N) \frac{\mathbb{1}_N}{N}. \tag{31}$$

As the weightages $s_i = (\lambda_i - \lambda_N)$ are non-negative, the above decomposition is necessarily unique. We note that Eq. (27) expresses P_N in terms of the eigenvalues of ρ. Using this, and after straightforward calculation, we obtain an expression for P_N solely in terms of the weightage of the pure parts given as

$$P_N = \sqrt{\frac{\sum_{i=1}^{N-1}\sum_{j=i+1}^{N} (\lambda_i - \lambda_j)^2}{(N-1)\left(\sum_{i=1}^{N} \lambda_i\right)^2}} = \sqrt{\frac{N\sum_{i=1}^{N-1} s_i^2 - \left(\sum_{i=1}^{N-1} s_i\right)^2}{N-1}}$$
$$= \sqrt{\left(\sum_{i=1}^{N-1} s_i\right)^2 - \frac{2N}{N-1}\sum_{i=1}^{N-1}\sum_{j=i+1}^{N-1} s_i s_j} \leqslant \sum_{i=1}^{N-1} s_i. \tag{32}$$

The above equation expresses the weightage of the pure part interpretation of P_N. Just as in the 2D case, we find that ρ can be generated by mixing together a completely mixed state and $N-1$ pure states in a particular proportion. However, the difference is that whereas $P_2 = s_1$ in the 2D case, for the ND case, we find $P_N \leq \sum_{i=1}^{N-1} s_i$. In other words, the total weightage of pure parts puts an upper bound on the intrinsic degree of coherence. Moreover, the bound is tight as in any ND space, and there exist states with only two non-zero eigenvalues. For such states, the bound is saturated, i.e., $P_N = \sum_{i=1}^{N-1} s_i$.

5. QUANTIFYING THE INTRINSIC DEGREE OF COHERENCE P_∞ OF INFINITE-DIMENSIONAL STATES

In this section, we extend P_N to the $N \to \infty$ limit to quantify the intrinsic degree of coherence P_∞ of infinite-dimensional states. The procedure is not quite as straightforward as computing the $N \to \infty$ limit of Eq. (2) due to the following reasons: first, from the expression for P_N, we note that in general, $\lim_{N\to\infty} P_N$ may not exist. This is because certain infinite-dimensional states can be non-normalizable, in which case $\mathrm{Tr}(\rho^2)$ can diverge [42]. Second, owing to the fact that N can take only integer values, even if $\lim_{N\to\infty} P_N$ exists, the generalization implicitly assumes the existence of a discrete or countable-infinite basis in the infinite-dimensional vector space. While this assumption is manifestly valid for the infinite-dimensional spaces spanned by the discrete OAM and photon number bases, its validity is not evident for the infinite-dimensional space spanned by the uncountable-infinite or continuous variable position and momentum bases. Here, we present rigorous derivation of P_∞ for infinite-dimensional states. We show that for any normalized infinite-dimensional state ρ in the OAM, photon number, and position and momentum bases, the expression for P_∞ is given by $P_\infty = \sqrt{\mathrm{Tr}(\rho^2)}$.

A. Orbital Angular Momentum and Angle Representations

We denote the OAM eigenstates as $|l\rangle$, where $l = -\infty, \ldots, -1, 0, 1, \ldots, \infty$, and the angle eigenstates as $|\theta\rangle$, where $\theta \in [0, 2\pi)$. Owing to the Fourier relationship between the OAM and angle observables [43], the eigenstates are related as

$$|l\rangle = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} e^{+il\theta} |\theta\rangle \mathrm{d}\theta, \tag{33a}$$

$$|\theta\rangle = \frac{1}{\sqrt{2\pi}} \sum_{l=-\infty}^{+\infty} e^{-il\theta}|l\rangle. \quad \textbf{(33b)}$$

We note that in contrast to finite-dimensional vectors, infinite-dimensional vectors may be non-normalizable. For instance, it is evident from Eq. (33b) that the angle eigenstate $|\theta\rangle$ is non-normalizable.

We now consider a state ρ written in the OAM basis as

$$\rho = \sum_{l=-\infty}^{+\infty} \sum_{l'=-\infty}^{+\infty} c_{ll'}|l\rangle\langle l'|. \quad \textbf{(34)}$$

We rewrite the state ρ of Eq. (34) in the limiting form

$$\rho = \lim_{D\to\infty} \sum_{l=-D}^{+D} \sum_{l'=-D}^{+D} c_{ll'}|l\rangle\langle l'|. \quad \textbf{(35)}$$

In essence, the above relation views the infinite-dimensional state ρ as the $D\to\infty$ limit of a $(2D+1)$-dimensional state residing in the finite state space spanned by the OAM eigenstates $|l\rangle$ for $l=-D, \ldots, -1, 0, 1, \ldots, D$, where D is an arbitrarily large but finite integer. We now use Eq. (2) to compute P_{2D+1} and evaluate $P_\infty = \lim_{D\to\infty} P_{2D+1}$, which yields

$$P_\infty = \lim_{D\to\infty} \sqrt{\frac{(2D+1)\sum_{l=-D}^{+D}\sum_{l'=-D}^{+D}|c_{ll'}|^2 - 1}{2D}}. \quad \textbf{(36)}$$

Now let us assume that ρ is normalized, i.e., $\mathrm{Tr}(\rho) = \sum_{l=-\infty}^{+\infty} c_{ll} = 1$. This implies that $\sum_{l=-\infty}^{+\infty}\sum_{l'=-\infty}^{+\infty}|c_{ll'}|^2 = \mathrm{Tr}(\rho^2) \le 1$. Under this condition, Eq. (36) evaluates to

$$P_\infty = \sqrt{\sum_{l=-\infty}^{+\infty}\sum_{l'=-\infty}^{+\infty}|c_{ll'}|^2} = \sqrt{\mathrm{Tr}(\rho^2)}. \quad \textbf{(37)}$$

The above equation can be used to evaluate P_∞ of a normalized state ρ. However, when ρ is non-normalizable, such as the angle eigenstate $\rho = |\theta\rangle\langle\theta|$ of Eq. (33b), the quantity $\mathrm{Tr}(\rho^2)$ diverges. In such cases, Eq. (37) cannot be used to compute P_∞.

We now use the basis invariance of P_∞ to derive its expression in terms of the angle representation of ρ. Using Eq. (33a) to substitute for $|l\rangle$ and $\langle l'|$ into Eq. (34), it follows that ρ has the angle representation

$$\rho = \int_0^{2\pi}\int_0^{2\pi} W(\theta,\theta')|\theta\rangle\langle\theta'|\mathrm{d}\theta\mathrm{d}\theta', \quad \textbf{(38)}$$

where the continuous matrix elements $W(\theta,\theta')$ are related to the coefficients $c_{ll'}$ as

$$W(\theta,\theta') = \frac{1}{2\pi}\sum_{l=-\infty}^{+\infty}\sum_{l'=-\infty}^{+\infty} c_{ll'}e^{+i(l\theta-l'\theta')}. \quad \textbf{(39)}$$

In the context of light fields, $W(\theta,\theta')$ is the angular coherence function, which quantifies the correlation between the field amplitudes at angular positions θ and θ' [44,45]. Assuming that ρ is normalized, we have $\mathrm{Tr}(\rho) = \int_0^{2\pi} W(\theta,\theta)\mathrm{d}\theta = 1$. Substituting Eq. (38) into Eq. (37), we obtain

$$P_\infty = \sqrt{\mathrm{Tr}(\rho^2)} = \sqrt{\int_0^{2\pi}\int_0^{2\pi}|W(\theta,\theta')|^2\mathrm{d}\theta\mathrm{d}\theta'}. \quad \textbf{(40)}$$

Equations (37) and (40) can be used to compute P_∞ of any normalized infinite-dimensional state in the OAM and angle representations.

B. Photon Number Representation

The photon number eigenstates $|n\rangle$, where $n = 0, \ldots, \infty$, span an orthonormal and complete basis in the infinite-dimensional Fock space. It is known that like OAM and angle, the photon number and optical phase are conjugate observables. However—owing to the fact that unlike the OAM eigenvalues, the photon number eigenvalues can take only non-negative integer values—the optical phase eigenstates in the infinite state space are not orthonormal, and therefore do not constitute a well-defined basis [46]. For our purposes, it is sufficient to restrict our attention to the photon number basis and compute P_∞ in an identical manner as we did previously for states in the OAM basis. We first consider a general state expressed in the photon number basis as

$$\rho = \sum_{n=0}^{\infty}\sum_{n'=0}^{\infty} a_{nn'}|n\rangle\langle n'|. \quad \textbf{(41)}$$

We rewrite the above state in the limiting form

$$\rho = \lim_{D\to\infty}\sum_{n=0}^{D}\sum_{n'=0}^{D} a_{nn'}|n\rangle\langle n'|, \quad \textbf{(42)}$$

where D is an arbitrarily large but finite positive integer. We then compute P_∞ of ρ by using Eq. (2) to compute P_{D+1} of a $(D+1)$- dimensional state in the limit $D\to\infty$ as

$$P_\infty = \lim_{D\to\infty}\sqrt{\frac{(D+1)\sum_{n=0}^{D}\sum_{n'=0}^{D}|a_{nn'}|^2 - 1}{D}}. \quad \textbf{(43)}$$

We assume that $\mathrm{Tr}(\rho) = \sum_{n=0}^{\infty} a_{nn} = 1$, which implies $\sum_{n=0}^{\infty}\sum_{n'=0}^{\infty}|a_{nn'}|^2 = \mathrm{Tr}(\rho^2) \le 1$. Under this condition, Eq. (43) reduces to the form

$$P_\infty = \sqrt{\sum_{n=0}^{\infty}\sum_{n'=0}^{\infty}|a_{nn'}|^2} = \sqrt{\mathrm{Tr}(\rho^2)}. \quad \textbf{(44)}$$

C. Position and Momentum Representations

We now consider infinite-dimensional states in the continuous-variable position and momentum representations. For conceptual clarity, we present our analysis for a 1D configuration space, which is labeled by the co-ordinate x. The corresponding canonical momentum space is labeled by the co-ordinate p. A general state ρ in the position basis is written as

$$\rho = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} G(x,x')|x\rangle\langle x'|\mathrm{d}x\mathrm{d}x'. \quad \textbf{(45)}$$

Similarly, in the momentum basis, ρ is given by

$$\rho = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \Gamma(p, p')|p\rangle\langle p'|\mathrm{d}p\mathrm{d}p'. \quad (46)$$

The continuous matrix elements $G(x, x')$ and $\Gamma(p, p')$ represent the cross-correlation functions in the position and momentum representations, respectively.

We recall that the expressions (37) and (44) for P_∞ of states in the OAM and photon number bases were derived by viewing the infinite-dimensional state as the infinite integer limit of a finite-dimensional state. As the dimensionality was constrained to take only integer values, the derivations implicitly depended on the fact that the OAM and photon number bases are discrete, and hence countable-infinite. However in the present case, both the position and the momentum bases are continuous, i.e., uncountable-infinite. Nevertheless, we now show that this issue can be circumvented by constructing a physically indistinguishable finite-dimensional state space for position and momentum variables. Our construction extensively draws on techniques developed previously by Pegg and Barnett for constructing finite-dimensional state spaces for the OAM angle [43] and photon number-optical phase [47,48] pairs of observables.

1. Construction of a Finite-Dimensional Space

We consider an arbitrarily large but finite region $[-p_{\max}, p_{\max}]$ in momentum space as depicted in Fig. 2. We sample $(2D+1)$ equally spaced momentum values p_j in this region, where $j = -D, \ldots, 0, \ldots, D$, with D also being arbitrarily large but finite. The spacing between consecutive values is $\Delta p = p_{\max}/D$, which is made arbitrarily close to zero. Using the $(2D+1)$ orthonormal eigenstates $|p_j\rangle$ corresponding to the momentum eigenvalues $p_j = j\Delta p$, we develop a consistent $(2D+1)$-dimensional state space for position and momentum. We will compute P_∞ for ρ by first computing P_{2D+1} of a $(2D+1)$-dimensional state and then taking the limit of $D \to \infty$ and $p_{\max} \to \infty$, subject to the condition that $1/\Delta p = D/p_{\max} \to \infty$.

To this end, we note that a momentum operator $\hat{p}$ must be a generator of translations in position space. Therefore, a position state $|x\rangle$ must satisfy [42]

$$\exp\left(-i\hat{p}\eta/\hbar\right)|x\rangle = |x+\eta\rangle. \quad (47)$$

If we define $|x_0\rangle$ as the state corresponding to the origin, then

$$|x\rangle = \exp\left(-i\hat{p}x/\hbar\right)|x_0\rangle. \quad (48)$$

Now, similarly, a position operator $\hat{x}$ must be a generator of translations in momentum space. This implies that

$$\exp\left(+ip_k\hat{x}/\hbar\right)|p_j\rangle = |p_{j+k}\rangle, \quad (49)$$

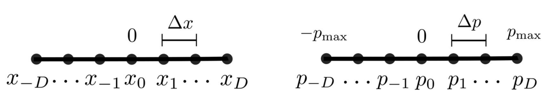

Fig. 2. In the finite state space, the position eigenvectors $|x_m\rangle$ for $m = -D, \ldots, 0, \ldots D$, and momentum eigenvectors $|p_j\rangle$ for $j = -D, \ldots, 0, \ldots, D$ span a finite $(2D+1)$-dimensional space.

where the translations are cyclic, such that $\exp(ip_1\hat{x}/\hbar)|p_D\rangle = |p_{-D}\rangle$. We now use the orthonormal states $|p_j\rangle$ and Eqs. (47) and (49) to derive the form of the corresponding position eigenstates in the $(2D+1)$-dimensional state space.

Let us suppose that $|x_0\rangle$ takes the general form

$$|x_0\rangle = \sum_{j=-D}^{+D} c_j|p_j\rangle. \quad (50)$$

Evaluating $\exp(+ip_k\hat{x}/\hbar)|x_0\rangle$ by using Eq. (49), we get

$$|x_0\rangle = \sum_{j=-D}^{+D} c_j|p_{j+k}\rangle. \quad (51)$$

Now since the above equation is true for all k, the coefficients c_j are necessarily independent of j, and upon normalization, they become $c_j = (1/\sqrt{2D+1})$. Using Eq. (48), we then obtain

$$|x\rangle = \sum_{j=-D}^{+D} \frac{e^{-ip_jx/\hbar}}{\sqrt{2D+1}}|p_j\rangle. \quad (52)$$

The inner product $\langle x|x'\rangle$ can therefore be written as

$$\langle x|x'\rangle = \sum_{j=-D}^{+D}\sum_{k=-D}^{+D} \frac{e^{+i(p_jx-p_kx')/\hbar}}{(2D+1)}\langle p_j|p_k\rangle$$
$$= \frac{1}{(2D+1)}\frac{\sin\left[(2D+1)(x-x')\Delta p/2\hbar\right]}{\sin\left[(x-x')\Delta p/2\hbar\right]}. \quad (53)$$

This implies that $\langle x|x'\rangle = 0$ only when $(x-x') = 2\pi\hbar n/\{(2D+1)\Delta p\}$, where n is a non-zero integer. This orthogonality condition allows us to select an orthonormal basis comprising the basis vectors $|x_m\rangle$ corresponding to the positions

$$x_m = \frac{2\pi m\hbar}{(2D+1)\Delta p} \qquad (m = -D, \ldots, 0, \ldots, D). \quad (54)$$

These $(2D+1)$ positions are equally spaced from x_{-D} to x_D with a spacing of $\Delta x = 2\pi\hbar/\{(2D+1)\Delta p\}$. We write the orthonormality and completeness relations for the basis vectors $|x_m\rangle$ and $|p_j\rangle$ as

$$\langle x_m|x_n\rangle = \delta_{mn}, \qquad \langle p_j|p_k\rangle = \delta_{jk}, \quad (55a)$$

$$\sum_{m=-D}^{+D}|x_m\rangle\langle x_m| = 1, \qquad \sum_{j=-D}^{+D}|p_j\rangle\langle p_j| = 1. \quad (55b)$$

Using Eqs. (52) and (54), we find that the basis vectors are related as

$$|x_m\rangle = \frac{1}{\sqrt{2D+1}}\sum_{j=-D}^{+D} e^{-i2\pi mj/(2D+1)}|p_j\rangle, \quad (56a)$$

$$|p_j\rangle = \frac{1}{\sqrt{2D+1}}\sum_{m=-D}^{+D} e^{+i2\pi mj/(2D+1)}|x_m\rangle. \quad (56b)$$

Thus, we have derived a finite-dimensional state space for position and momentum, which is depicted schematically in Fig. 1. In order to prove that the finite state space is physically consistent, we must show that the commutator $[\hat{x}, \hat{p}]$ in this space is physically indistinguishable from the improper commutation relation $[\hat{x}, \hat{p}] = i\hbar$. To this end, we note that $\hat{x} = \sum_{m=-D}^{+D} x_m |x_m\rangle\langle x_m|$ and $\hat{p} = \sum_{j=-D}^{+D} p_j |p_j\rangle\langle p_j|$. Using these expressions, we find that the commutator $[\hat{x}, \hat{p}]$ has the following matrix elements:

$$\langle x_m|[\hat{x}, \hat{p}]|x_n\rangle = \frac{2\pi\hbar(m-n)}{(2D+1)^2} \sum_{j=-D}^{+D} j e^{i2\pi(m-n)j/(2D+1)}, \tag{57a}$$

$$\langle p_j|[\hat{x}, \hat{p}]|p_k\rangle = \frac{2\pi\hbar(k-j)}{(2D+1)^2} \sum_{m=-D}^{+D} m e^{-i2\pi(j-k)m/(2D+1)}. \tag{57b}$$

We notice that the diagonal elements $\langle x_m|[\hat{x}, \hat{p}]|x_m\rangle$ and $\langle p_j|[\hat{x}, \hat{p}]|p_j\rangle$ are all zero. As a result, the trace of $[\hat{x}, \hat{p}]$ is zero, as expected for any commutator of finite-dimensional operators. We evaluate the above Eq. (57) in the limit $D \to \infty$ using Mathematica [49], and simplify to obtain

$$[\hat{x}, \hat{p}] = \lim_{D\to\infty} i\hbar \left[1 - (2D+1)\left|x_{\left(D+\frac{1}{2}\right)}\right\rangle\!\left\langle x_{\left(D+\frac{1}{2}\right)}\right|\right], \tag{58a}$$

$$= \lim_{D\to\infty} i\hbar \left[1 - (2D+1)\left|p_{\left(D+\frac{1}{2}\right)}\right\rangle\!\left\langle p_{\left(D+\frac{1}{2}\right)}\right|\right]. \tag{58b}$$

We find that when the expectation value of $[\hat{x}, \hat{p}]$ is evaluated for any physical state, the contributions from the second term in the above expressions asymptotically vanish. In this limit, we recover the usual commutator $[\hat{x}, \hat{p}] = i\hbar$ for infinite-dimensional operators. Thus, we have constructed a consistent finite-dimensional state space for position and momentum.

2. Derivation of the Expression for P_∞

We write the state ρ from Eq. (45) in the position basis of the finite-dimensional state space as

$$\rho = \lim_{D\Delta x\to\infty} \lim_{\Delta x\to 0} \sum_{m=-D}^{+D} \sum_{n=-D}^{+D} \bar{G}_{x_m x_n} |x_m\rangle\langle x_n|. \tag{59}$$

Similarly, ρ can be written in the momentum basis as

$$\rho = \lim_{D\Delta p\to\infty} \lim_{\Delta p\to 0} \sum_{j=-D}^{+D} \sum_{k=-D}^{+D} \bar{\Gamma}_{p_j p_k} |p_j\rangle\langle p_k|. \tag{60}$$

As ρ is normalized, we have $\sum_{m=-D}^{+D} \bar{G}_{x_m x_m} = \sum_{j=-D}^{+D} \bar{\Gamma}_{p_j, p_j} = 1$. We can compute P_∞ for ρ by first computing P_{2D+1} in terms of $\bar{G}_{x_m x_n}$ and $\bar{\Gamma}_{p_j p_k}$, and then evaluating its limiting value as $D \to \infty$ and $p_{\max} \to \infty$, subject to the constraint $D/p_{\max} \to \infty$. These limits together ensure that $\Delta x \to 0$ and $\Delta p \to 0$, such that $D\Delta x \to \infty$ and $D\Delta p \to \infty$. Thus, we can compute P_∞ in terms of $\bar{G}_{x_m x_n}$ as

$$P_\infty = \lim_{D\Delta x\to\infty} \lim_{\Delta x\to 0} \sqrt{\frac{2D+1}{2D}\left[\sum_{m,n} |\bar{G}_{x_m x_n}|^2 - \frac{1}{2D+1}\right]}. \tag{61}$$

Similarly, in terms of $\bar{\Gamma}_{p_j p_k}$, we have

$$P_\infty = \lim_{D\Delta p\to\infty} \lim_{\Delta p\to 0} \sqrt{\frac{2D+1}{2D}\left[\sum_{j,k} |\bar{\Gamma}_{p_j p_k}|^2 - \frac{1}{2D+1}\right]}. \tag{62}$$

In order to derive the form of P_∞ in terms of $G(x, x')$ and $\Gamma(p, p')$, we must obtain the relation of these continuous functions to their discrete counterparts $\bar{G}_{x_m x_n}$ and $\bar{\Gamma}_{p_j p_k}$, respectively. Now if ρ is a physical state, then $G(x, x')$ and $\Gamma(p, p')$ must be continuous integrable functions normalizable to unity. Thus, the relation of $G(x, x')$ to $\bar{G}_{x_m x_m}$, and that of $\Gamma(p, p')$ to $\bar{\Gamma}_{p_j p_k}$, must be such that $\sum_{m=-D}^{+D} \bar{G}_{x_m x_m} = \sum_{j=-D}^{+D} \bar{\Gamma}_{p_j, p_j} = 1$ should imply $\int_{-\infty}^{+\infty} G(x, x)\mathrm{d}x = \int_{-\infty}^{+\infty} \Gamma(p, p)\mathrm{d}p = 1$. We now consider the relations

$$G(x_m, x_n) = \lim_{D\Delta x\to\infty} \lim_{\Delta x\to 0} \bar{G}_{x_m x_n}/\Delta x, \tag{63a}$$

$$\Gamma(p_j, p_k) = \lim_{D\Delta p\to\infty} \lim_{\Delta p\to 0} \bar{\Gamma}_{p_j p_k}/\Delta p. \tag{63b}$$

Substituting the above relations into $\sum_{m=-D}^{+D} \bar{G}_{x_m x_m} = \sum_{j=-D}^{+D} \bar{\Gamma}_{p_j, p_j} = 1$ yields $\lim_{D\Delta x\to\infty} \lim_{\Delta x\to 0} G(x_m, x_m)\Delta x$ and $\lim_{D\Delta p\to\infty} \lim_{\Delta p\to 0} \Gamma(p_j, p_j)\Delta p = 1$. These summations are equivalent to the integral relations $\int_{-\infty}^{+\infty} G(x, x)\mathrm{d}x = \int_{-\infty}^{+\infty} \Gamma(p, p)\mathrm{d}p = 1$, which implies that Eq. (63) is correct. Upon substituting Eq. (63a) into Eq. (61), and Eq. (63b) into Eq. (62) and simplifying, we obtain

$$P_\infty = \lim_{D\Delta x\to\infty} \lim_{\Delta x\to 0} \sqrt{\sum_{m,n=-D}^{+D} |G(m\Delta x, n\Delta x)|^2 \Delta x \Delta x},$$

$$P_\infty = \lim_{D\Delta p\to\infty} \lim_{\Delta p\to 0} \sqrt{\sum_{j,k=-D}^{+D} |\Gamma(j\Delta p, k\Delta p)|^2 \Delta p \Delta p}.$$

The above equations can be expressed in integral form as [50]

$$P_\infty = \sqrt{\iint_{-\infty}^{+\infty} |G(x, x')|^2 \mathrm{d}x\mathrm{d}x'} = \sqrt{\mathrm{Tr}(\rho^2)}, \tag{64a}$$

$$P_\infty = \sqrt{\iint_{-\infty}^{+\infty} |\Gamma(p, p')|^2 \mathrm{d}p\mathrm{d}p'} = \sqrt{\mathrm{Tr}(\rho^2)}. \tag{64b}$$

Moreover, in terms of the Wigner function representation $W(x, p) = (1/(\pi\hbar)) \int_{-\infty}^{+\infty} \langle x + y|\hat{\rho}|x - y\rangle e^{-2ipy/\hbar}\mathrm{d}y$ of ρ [51], the measure P_∞ can be expressed as

$$P_\infty = \sqrt{\mathrm{Tr}(\rho^2)} = \sqrt{2\pi\hbar \iint_{-\infty}^{+\infty} W^2(x, p)\mathrm{d}x\mathrm{d}p}. \tag{65}$$

We note that the form of P_∞ in Eq. (64a) is identical to a measure known as the "overall degree of coherence" that was introduced and employed by Bastiaans for characterizing the spatial coherence of partially coherent fields in a complete manner [52,53]. Here, we have derived the measure for general classical and quantum states in the position and momentum representations from an entirely distinct perspective.

6. CONCLUSION AND DISCUSSION

In the context of 2D partially polarized electromagnetic fields, the basis-independent degree of polarization P_2 can be used to quantify the intrinsic degree of coherence of 2D states. The measure P_2 has six known interpretations: (i) it is the Frobenius distance between the state and the identity matrix, (ii) it is the norm of the Bloch vector representing the state, (iii) it is the distance to the center of mass in a configuration of point masses, (iv) it is the maximum of the degree of coherence, (v) it is the visibility in a polarization interference experiment, and (vi) it is equal to the weightage of the pure part of the state. By generalizing the first three interpretations, past studies derived analogous expressions for the intrinsic degree of coherence P_N of ND states. In this paper, we extended the concepts of visibility, degree of coherence, and weightage of the pure part to ND states, and showed that P_2 generalizes to P_N with respect to these interpretations as well. While other yet-to-be-discovered interpretations may still exist, we showed that P_N has all the known interpretations of P_2 and can therefore be regarded as the intrinsic degree of coherence of ND states. Finally, we extended the formulation of P_N to the $N \to \infty$ limit and quantified the intrinsic degree of coherence P_∞ of infinite-dimensional states in the OAM, photon number, and position and momentum representations.

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